TACTICS AND STRATEGIES FOR TACKLING THE CAMBRIDGE SIXTH TERM EXAMINATION PAPERS

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Abstract

School mathematics is taught in a way which responds to the requirements of examination board specifications. The material is presented mainly as calculation techniques and mathematical methods for answering closed questions. The Sixth Term Entrance Papers (STEP) aim to assess potential for studying Mathematics at a higher level by testing problem solving and a wider awareness of meaning and context. This is a fresh approach to the subject but, like all things, it can be taught. Mostly this is a case of filling in the gap between the standard school material (mathematical methods) and real mathematics. These techniques and strategies are assembled to give the student a number of tools to open up problems and to develop a wider appreciation of meaning which will help students problem solve and answer STEP questions.

1 PROOF

A growing sense of formal proof is important to the prospective STEP student. Much useful foundational material is available in [1]. Whilst this book is meant as preparation for university courses it actually fairly deep and essentially covers foundational material taught in the first year of university courses.

In STEP questions we have to recognise the difference between the varying degrees of rigour required by the following instructions often given in STEP questions:

- **Explain Briefly**. An outline showing understanding.
- **Show that** A demonstration of a result using valid and understood techniques such as algebra.
- **Prove** A heavily loaded statement requiring a more formal and exhaustive answer.

What follows now is a summary of types and techniques of proof that are frequently required in STEP.

1.1 Exhaustive Argument

. In formal proof it is crucial to examine all cases which arise in a given situation in a structured and exhaustive fashion. It is easy to ignore or forget cases and STEP examiners will be on the look out for the greater awareness shown by a student who thoroughly deals with all eventualities in an argument.

1.2 If and Only If

If and only if (symbol $\Leftrightarrow$) is a very different prospect from if (symbol $\Rightarrow$).

For example, the statement '$x > -2 \Rightarrow x > -5$' is true but the statement '$x > -2 \Leftrightarrow x > -5$' is false because $x > -5$ is not sufficient to mean that $x > -2$, x might be -3. If and only if proofs generally require two parts an 'if' or $\Rightarrow$ part and an 'only if' or $\Leftarrow$ part. Sometimes elegant arguments involving identities (trigonometric or algebraic) can be constructed using $\Leftrightarrow$ arguments at each point.

1.3 Contradiction

If proving $A \Rightarrow B$ assume that A is not true and follow logical reasoning to obtain a contradiction; this proves that A must be true. The loose reasoning is that if A is not true then it would lead to a farcical situation hence A must be true. Not convinced? Some are not, but for more abstract details refer to [1] where you learn that you can address the formal logic aspects of proof using Venn diagrams and truth details. Such research should appeal to a Maths and Philosophy student.

An example of proof by contradiction:

Example 1.1 *$\sqrt{2}$ is irrational.*

Suppose that $\sqrt{2}$ is rational i.e. the statement is not true. Then it is possible to find $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \neq 0$ such that $\sqrt{2} = \frac{a}{b}$ and a and b have no common factor (otherwise cancel down ..). Then $2 = \frac{a^2}{b^2}$ and so $a^2 = 2b^2$. So a must be even (a square can only be even if the number being squared is also even). This means that there exists $p \in \mathbb{Z}$ such that $a = 2p$ which means that $b = 4p^2$ and hence b is even too. Thus if the proposition is not true both a and b are even which contradicts the fact that they have no common factor. Hence the proposition is true and $\sqrt{2}$ is irrational.

1.4 Induction

Mathematical induction is covered in FP3 and will probably restricted to STEP II and III. Nevertheless, it can be studied with profit by STEP I students.

When first meeting induction some students are left with a feeling that there is some trickery afoot and that they are getting something for nothing. Such an instinct is spot on because Mathematical Induction is an AXIOM, that is to say a fact which we take on trust without proof. This axiom amounts to believing that we can count in natural numbers (e.g. 1, 2, 3, ...) indefinitely and up to any required natural number. A more logical statement follows:

Let S be a set which only includes natural numbers i.e. if $n \in\ S$ then $n \in \mathbb{N}$. If $0 \in S$ and $(k \in S) \Rightarrow k+1 \in S$ then the axiom of induction states that $S = \mathbb{N}$.

Because it is about counting induction can only be applied to statements which involve whole positive numbers. The way that this is applied to proof is illustrated by the following easy example.

Example 1.2

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1) \textit{ for all } n \in \mathbb{N}$$

The background (and unstated idea) is that we set up a set of natural numbers S for which the above statement is true. There are three steps in any proof by induction and these need to be laid out formally before the magic of the axiom of induction can be invoked. The first task is to prove that $0 \in S$.

***Prove true for** $n = 0$. Now $0 = \frac{1}{2}0(0+1)$, (hence $0 \in S$).*

***Assume true for** $n = k$. Sometimes called the Inductive Hypothesis (IH). We assume that*

$$1 + 2 + 3 + \ldots + k = \frac{1}{2}k(k+1)$$

***Prove true of** $n = k + 1$ **by assuming the IH.**.*

Now,

$$\begin{aligned}
1 + 2 + 3 + \ldots + k + (k+1) &= \overbrace{1 + 2 + 3 + \ldots + k}^{=\frac{1}{2}k(k+1)} + (k+1) \\
&= \frac{1}{2}k(k+1) + (k+1) \textit{ by IH} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{1}{2}(k+1)(k+2) \textit{ our required result}
\end{aligned}$$

hence true for $n = k + 1$. *(We have proved that* $(k \in S) \Rightarrow k + 1 \in S$.*)*

Hence by Induction (axiom of induction implies $S = \mathbb{N}$*) the statement is true for all* $n \in \mathbb{N}$.

The astute student realises that this is not all there is to induction. The axiom stated above can be re-written in a stronger form: the step at which we prove the statement true for $n = k + 1$ does not necessarily need to refer to an IH involving $n = k$ it could refer to any $n < k + 1$. This is called strong induction. A statement of the axiom here would be

Let S be a set which only includes natural numbers i.e. if $n \in S$ then $n \in \mathbb{N}$. If $0 \in S$ and $(k \in S$ with $k < n) \Rightarrow n \in S$ then the axiom of strong induction states that $S = \mathbb{N}$.

2 Division by Zero

When working long and involved algebraic arguments it is some times easy to divide by zero without realising it.

Example 2.1 *Solve*

$$x \sin x + \sin x = x + 1$$

factorising gives

$$(x + 1) \sin x = x + 1$$

at this point there may be a temptation to divide both sides of the equation by $(x + 1)$. *This must be treated with care:* $(x + 1)$ *could be zero (at* $x = -1$*) in which case the laws of algebra do not permit a division. Division by* $(x + 1)$ *can only be carried out if the case* $(x + 1) = 0$ *is dealt with at this point. One solution to the equation is* $x = -1$, *if* $x \neq -1$ *divide by* $(x + 1)$ *and continue to solve* $\sin x = 1$. *Division by zero here costs one solution to the equation. The best way to proceed is not to divide by any function unless it can be proved that the function cannot be zero. The solution to our example runs as follows*

$$\begin{aligned}
(x + 1) \sin x = x + 1 &\Leftrightarrow (x + 1) \sin x - (x + 1) = 0 \\
&\Leftrightarrow (x + 1)(\sin x - 1) = 0 \\
&\Leftrightarrow \textit{ either } x + 1 = 0 \textit{ or } \sin x - 1 = 0 \\
&\Leftrightarrow x = -1 \textit{ or } x = 2n\pi + \frac{\pi}{2}
\end{aligned}$$

Note the use of general solution of trigonometric equations described later.

3 Functions and Curve Sketching

3.1 Functions

In most cases at school we consider functions which produce well behaved graphs with smooth, continuous curves. Polynomials, trigonometric functions, exponential functions and logarithms all fall into this category. In some cases curves have discontinuities for example

$$\begin{aligned} y &= \frac{1}{x-4} \\ y &= \sec x \end{aligned}$$

The list could contain more complicated algebraic fractions which have multiple asymptotes. What links these curves is that they are all, in some sense, smooth: at any point one can fit a tangent to them They are **differentiable** at all points except at their discontinuities.

Other functions introduce kinks and sharp bits where it is not possible to fit a tangent. For example:

$$y = |x^2 - 4|$$

which, whilst continuous throughout, has an sharp bit at $x = \pm 2$ where a tangent cannot be drawn. These functions are not differentiable.

Mathematicians introduce a range of other functions for various purposes, these generate interesting graphs, but they are very different from the smooth differentiable functions we are used to dealing with. For example:

$$y = [x]$$

Where $[x]$ is called the integer part functions and,

$$[x] = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ z & \text{if } z \in \mathbb{Z} \text{ such that } z \text{ is the greatest integer less than x} \end{cases}$$

Such functions behave differently from the functions you are used to and students must be alert to their qualities when they are introduced.

3.2 Curve Sketching

A note about curve sketching. No calculators, graphical or otherwise, are allowed in STEP. Students are on their own when it comes to sketching graphs and understanding curves. The skill of curve sketching is often tested at university interviews.

Learning the shape of the graphs of standard functions is essential, as is the effects of transformations on these graphs (shifts and stretches on the x and y axes). When knowledge fails a systematics technique is needed. One such technique is **ZISTA**, an acronym where:

Z = ZEROS or intercepts on both axes
let $x = 0$ and then solve for y then let $y = 0$ and solve for x

I = INFINITIES or what happens to the function for large values of x or y, are there any oblique asymptotes?

S = SIGN CHANGES, understand the sign changes of the function before and after asymptotes and zeros

T = TURNING POINTS, differentiate and analyse maximums and minimums and points of inflection

A = ASYMPTOTES, look for vertical, horizontal and oblique asymptotes

After any such comprehensive analysis you will have enough information to sketch most functions.

4 Inequalities

4.1 When Proving A>B consider A-B

When proving $A > B$ it is almost always easier to prove $A - B > 0$.

Example 4.1 *Prove $a^2 + b^2 \geqslant 2ab$ for all $a, b \in \mathbb{R}$.*

$$\begin{aligned} a^2 + b^2 - 2ab &= a^2 - 2ab + b^2 \\ &= (a-b)^2 \geqslant 0 \textit{ since it is a square} \end{aligned}$$

and the result is proved.

4.2 Critical Points

Apply this technique when trying to solve inequalities of the form:

$$P(x) = \frac{f(x)}{g(x)}$$

The question amounts to determining the sign of $P(x)$. In many ways it is the same problem as curve sketching already described. This technique is all about analysing **critical points** which are defined as those values for x at which the function $P(x)$ changes sign: the S in ZISTA. For 'well behaved' functions which are piecewise smooth and continuous sign changes will only occur at values of x where there is a ZERO (x-intercept) or an ASYMPTOTE.

ZEROS are found by solving $f(x) = 0$.

ASYMPTOTES are found by solving $g(x) = 0$.

Once all critical values have been found you can systematically test the sign of the function in between the values.

Example 4.2 *Solve*

$$\frac{\sin\theta + 1}{\cos\theta} \leqslant 1 \text{ for } 0 \leqslant \theta \leqslant 2\pi$$

When solving $A \geqslant B$ it is almost always easier to look at $A - B \geqslant 0$.

So

$$\begin{aligned}\frac{\sin\theta + 1}{\cos\theta} \leqslant 1 \quad &\Leftrightarrow \quad \frac{\sin\theta + 1}{\cos\theta} - 1 \leqslant 0 \\ &\Leftrightarrow \quad \frac{\sin\theta - \cos\theta + 1}{\cos\theta} \leqslant 0 \\ &\Leftrightarrow \quad \frac{\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) + 1}{\cos\theta} \leqslant 0\end{aligned}$$

ASYMPTOTES at $\cos\theta = 0$ i.e. $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

ZEROS at $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) + 1 = 0$

Now,

$$\begin{aligned}\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) + 1 = 0 \quad &\Leftrightarrow \quad \sin\left(\theta - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \\ &\Leftrightarrow \quad \theta - \frac{\pi}{4} = -\frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{7\pi}{4} \\ &\Leftrightarrow \quad \theta = 0 \text{ or } \frac{3\pi}{2} \text{ or } 2\pi\end{aligned}$$

Hence the ZEROS are at $\theta = 0$ or $\frac{3\pi}{2}$ or 2π and critical points are $\theta = 0$ or $\frac{3\pi}{2}$ or 2π

Now we can analyse the sign changes of

$$\frac{\sqrt{2}\sin(\theta - \frac{\pi}{4}) + 1}{\cos\theta}$$

This can be done systematically using a table:

	$0 < \theta < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$	$\frac{3\pi}{2} < \theta < 2\pi$
$\sqrt{2}\sin(\theta - \frac{\pi}{4}) + 1$	+	+	-
$\cos\theta$	+	-	+
$\frac{\sqrt{2}\sin(\theta - \frac{\pi}{4}) + 1}{\cos\theta}$	+	-	-

Hence the inequalitity is solved for the following values of θ: $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$. (Some more work needs to be done to consider the point $\theta = 2\pi$).

4.3 Inequalities and Hidden Negative Quantities

4.4 Squaring Inequalities

Note that $A < B \nRightarrow A^2 < B^2$, consider $A = -5$ and $B = 2$.

Of course $A > 0, B > 0 and A < B \Rightarrow A^2 < B^2$.

5 Trigonometry

5.1 General Solution of Trigonometric Equations

- **Sine** If $\sin\theta = a$ then $\theta = n\pi + (-1)^n\theta_p$ where θ_p is the priniple value of $\arcsin a$.
- **Cosine** If $\cos\theta = a$ then $\theta = 2n\pi + \pm\theta_p$ where θ_p is the priniple value of $\arccos a$.
- **Tangent** If $\tan\theta = a$ then $\theta = n\pi + \theta_p$ where θ_p is the priniple value of $\arctan a$.

5.2 Sine is Odd, Cosine is Even

Examination of the graphs for Sine and Cosine reveal the following simple facts. These straight forward facts can be lost in the middle of intense trigonometric algebra and yet they can simplify things a great deal.

- Sine is an ODD Function: i.e. $\sin(-\theta) = -\sin(\theta)$
- Cosine is an EVEN Function: i.e. $\cos(-\theta) = \cos(\theta)$
- Tangent is an ODD Function: i.e. $\tan(-\theta) = -\tan(\theta)$

5.3 Transformation of Sine to Cosine

It is often hand to recognise the following transformationships:

- $\sin(\theta) \equiv \cos(\theta - \frac{\pi}{2})$
- $\sin(\theta + \frac{\pi}{2}) \equiv \cos(\theta)$

6 Algebra

6.1 Symmetric Polynomials

The connections between roots and coefficients in polynomials once was standard algebra in Further Mathematics. The subject matter motivated abstract algebra in the first place and is still central to the study of invariants at higher levels. STEP questions frequently play on the relationships found here.

Symmetric polynomials arise in a natural way in the connection between roots and coefficients in polynomials:

$$\begin{aligned}(x-\alpha)(x-\beta) &= x^2-(\alpha+\beta)x+\alpha\beta\\(x-\alpha)(x-\beta)(x-\gamma) &= x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma\end{aligned}$$

If,

$$x^2+bx+c=(x-\alpha)(x-\beta)$$

then,

$$\begin{aligned}b &= -(\alpha+\beta)\\c &= \alpha\beta\end{aligned}$$

.

If,

$$x^3+bx^2+cx+d=(x-\alpha)(x-\beta)(x-\gamma)$$

then,

$$\begin{aligned}b &= -(\alpha+\beta+\gamma)\\c &= (\alpha\beta+\alpha\gamma+\beta\gamma)\\d &= -\alpha\beta\gamma\end{aligned}$$

Thus the coefficients of a polynomial are polynomials of their roots. Note that the polynomials expressing b, c and d are unchanged if α, β and γ are permuted, such polynomials are said to be symmetric.

The elementary symmetric polynomials of degree 2 are:

$$\alpha + \beta$$
$$\alpha\beta$$

and the elementary symmetric functions of degree 3 are:

$$\alpha + \beta + \gamma$$
$$\alpha\beta + \alpha\gamma + \beta\gamma$$
$$\alpha\beta\gamma$$

Other polynomials are symmetric but not elementary:

$$ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2$$

Newton proved that every symmetric polynomial can be written uniquely as a polynomial in the elementary symmetric functions.

Thus

$$ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2 = (a + b + c)(ab + ac + bc) - 3abc$$

Using these relationships in algebraic questions can be useful.

6.2 Advanced Use of the Factor Theorem

Some STEP questions look like they involve heavy algebra when they don't. Students are used to doing algebra with x and y with things like completing the square. These techniques have wider application. Consider the following use of the factor theorem to quickly factorise an expression found in a STEP questions.

Example 6.1 *Factorise*

$$abc - (a + b + c)(ab + ac + bc)$$

At first sight a student might want to expand out gather like terms and hope that a factorisation will become apparent. However, consider what happens in this expression if you let $a = -b$. You then have:

$$-b^2c - c(-b^2 - bc + bc)$$

which expands thus,

$$-b^2c + +b^2c + bc - bc = 0$$

invoking the factor theorem we can conclude that $(a+b)$ must be a factor of the expression. Also note that $abc-(a+b+c)(ab+ac+bc)$ is symmetric hence if $(a+b)$ is a factor then so is $(b+c)$ and $(a+c)$ purely by symmetry. Also, $abc-(a+b+c)(ab+ac+bc)$ has degree 3 in that every term is a product of three variables so the only possibilities for the factorisation are: $(a+c)(a+b)(b+c)$ or $-(a+c)(a+b)(b+c)$. A little thought reveals that,

$$abc-(a+b+c)(ab+ac+bc) = -(a+c)(a+b)(b+c)$$

and the job is done painlessly.

6.3 Algebraic Identities

$$(1-x^n) = (1-x)(1+x+x^2+\ldots+x^{n-1}$$

$$\frac{1}{(1-x)^n} = 1-x+x^2-x^3+x^4\ldots$$

Only converges for $|x|<1$.

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2 \text{ or } (ac+bd)^2+(ad-bc)^2$$

The product of two sums of squares is itself the sum of two squares. Note relationship with complex numbers: $|a+bi||c+di| = |(a+bi)(c+di)|$.

6.4 Transforming Equations

Can solve

$$\mathrm{f}(x) = 4x^3-4x-\cos 3\alpha = 0$$

to get roots in terms of $\cos\alpha$ and $\sin\alpha$. [See STEP I 2015.]

Need to solve

$$\mathrm{g}(y) = y^3-4y-\sqrt{2} = 0$$

Is there a 'C3' functional transformation which links the two functions? Is there "a" and "b" such that:

$$\mathrm{f}(x) = b\mathrm{g}(ax)$$

If so, $\mathrm{f}(\alpha)=0$ would imply that $\mathrm{g}(a\alpha)=0$ and that the roots, α of f transform to roots $a\alpha$ of g. Now,

$$4x^3-4x-\cos 3\alpha = b(ax)^3-4bax-b\sqrt{2}$$

In which case $ba^3=4$ and $3ba=3$ with solution $a=2$, $b=\frac{1}{2}$ and $\cos 3\alpha = \frac{1}{\sqrt{2}}$

$$\frac{1}{2}\mathrm{g}(2x) = 0$$

$$\implies \frac{1}{2}(2x)^3 - \frac{1}{2}8x - \frac{1}{2}\sqrt{2} = 0$$
$$\implies 4x^3 - 4x - \frac{1}{\sqrt{2}} = 0$$

Solve for x with $\cos 3\alpha = \frac{1}{\sqrt{2}}$ then multiply each root by 2 to get corresponding roots of y.

7 Useful Results in Co-ordinate Geometry

7.1 Heron's Formula

The area of a triangle whose sides have lengths a, b and c is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter

$$s = \frac{a+b+c}{2}$$

Proof. By cosine rule. Labelling sides and angles in the conventional way

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Now,

$$\begin{aligned} \sin C &= \sqrt{1-\cos^2 C} \\ &= \frac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab} \end{aligned}$$

So,

$$\begin{aligned} A &= \frac{1}{2}ab\sin C \\ &= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2} \\ &= \frac{1}{4}\sqrt{(2ab-(a^2+b^2-c^2)(2ab+(a^2+b^2-c^2)} \\ &= \frac{1}{4}\sqrt{(c^2-(a-b)^2)((a+b)^2-c^2)} \\ &= \frac{1}{4}\sqrt{(c-(a-b))(c+(a-b))((a+b)-c)((a+b)+c)} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

a kind of symphony in the difference of two squares.

7.2 Distance From a Point to a Line

The distance of the point (x_0, y_0) to the line $ax + by + c = 0$ is given by

$$d = \frac{|ax_0 + b_0 + c|}{\sqrt{a^2 + b^2}}$$

Proof. By coordinate geometry; seting up parallel lines and using Pythagoras (quite long - to be included later. Alternatively by vectors and scalar product similar to FP3 proof of distance of a point in 3-d to a plane.

8 Dualities

A STEP problem can often run aground and get stuck fast in the mud. Often this is because the student has been seduced into a head on attack using the obvious tools; I believe that the problems are designed this was. The answer comes from taking a fresh approach, turning the problem upside down and looking at it from a different perspective. This is where my list of dualities comes in handy, of course once you become mathematically aware these things will become self evident.

8.1 algebra-geometry

This is the most basic and most general one. When struggling with heavy algebra is it better to consider curves and shapes instead?

8.2 Integration: antiderivative - area

When struggling with rules of integration for a definite integral is it better to view the thing as an area?

8.3 Max/Min: differentiation - algebra

Questions which appear to be about differential calculus and finding max/min might turn out to be easier with an algebraic (e.g. completing the square) treatment.

9 Combinatorics

The study of systematic counting.

References

[1] Stuart and Tall, **The Foundations of Mathematics**, ?????